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Artin–Tits groups with CAT(0) Deligne complex

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Abstract

Let (A, S) be an Artin–Tits system, and A_X be the standard parabolic subgroup of A generated by a subset X of S. Under the hypothesis that the Deligne complex has a CAT(0) geometric realization, we prove that the normalizer and the commensurator of A_X in A are equal. Furthermore, if A_X is of spherical type, these subgroups are the product of A_X with the quasi-centralizer of A_X in A. For two-dimensional Artin–Tits groups, the result still holds without any sphericality hypothesis on X. We explicitly describe the elements of this quasi-centralizer.

MSC: 20F36; 57M07; 53C23

Introduction

Artin–Tits groups are a natural generalization of braid groups. They are defined by presentations involving relations similar to the standard braid relations but with length not necessarily equal to 2 or 3. The properties of general Artin–Tits groups remain mysterious although some special families, like the family of spherical type Artin–Tits groups (see Section 1 for a definition), are better known. An Artin–Tits group has a natural family of subgroups, namely the so-called parabolic subgroups: a standard parabolic subgroup is a subgroup generated by a subset of the distinguished generating set; a parabolic subgroup is a subgroup that is conjugated to a standard parabolic subgroup. A standard parabolic subgroup is itself (and canonically) an Artin–Tits group [16]. It turns out that the study of the family of parabolic subgroups is crucial for understanding the whole group. Significant results have been proved by Van der Lek in [16] but a lot of questions remain open.

In the last few years larger families of Artin–Tits groups have been studied and are now well understood, especially the family of spherical type Artin–Tits groups (see [9,10,13,15]) and the family of FC type Artin–Tits groups (see [11]). In these cases, both combinatorial and geometrical methods have been successfully applied. Here we address three specific properties of Artin–Tits groups involving the normalizer, the parabolic subgroups and the category of ribbons. These properties were known to hold in some special cases, and we conjecture they always do. The aim of this paper is to establish the three properties for new cases.

The first property is concerned with the relation between an Artin–Tits group and its parabolic subgroups. We denote by $\mathrm{Com}_{A_S}(A_X)$, $\mathrm{N}_{A_S}(A_X)$ and $\mathrm{QZ}_{A_S}(X)$ the commensurator, the normalizer and the quasi-centralizer,

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respectively, of the standard parabolic subgroup A_X in the Artin–Tits group A_S (see the next section for a precise definition).

Definition 1. Let (A_S, S) be an Artin–Tits system, and X be a subset of S. We say that the standard parabolic subgroup A_X has *Property* $(\star)$ when

$$\mathrm{Com}_{A_S}(A_X) = \mathrm{N}_{A_S}(A_X) = A_X \cdot \mathrm{QZ}_{A_S}(X). \tag{$\star$}$$

We say that the group A_S has *Property* $(\star)$ if all its standard parabolic subgroups have Property $(\star)$.

Some inclusions in $(\star)$ are obvious and Property $(\star)$ says that the commensurator (and thereby the normalizer) of a parabolic subgroup is small.

The second property describes the inclusion of a parabolic subgroup in another.

Definition 2. Let (A_S, S) be an Artin–Tits system, and X, Y be included in S. We say that the pair (X, Y) has *Property* $(\star\star)$ if for every g in A_S, we have the implication

$$gA_Xg^{-1} \subseteq A_Y \Rightarrow \exists h \in A_Y \exists Z \subseteq Y \text{ such that } gA_Xg^{-1} = hA_Zh^{-1}. \tag{$\star\star$}$$

We say that the group A_S has *Property* $(\star\star)$ if all pairs (X, Y) with X, Y included in S have Property $(\star\star)$.

In other words, if A_S has Property $(\star\star)$, then every parabolic subgroup of A_S included in a standard parabolic subgroup A_Y is a parabolic subgroup of A_Y.

The third property is concerned with the problem of conjugacy between two parabolic subgroups and it involves the category of ribbons. In the case of the braid group, there exists a natural notion of ribbon associated with a pair of strings that move together along the braid. The notion was used in [9] to describe the quasi-centralizers and subsequently generalized from a combinatorial viewpoint in [13] and [10]. Roughly speaking, the objects of the category of conjugators $\mathrm{Conj}(S)$ are parabolic subgroups of A_S and the morphisms are the elements of A_S that conjugate a standard parabolic subgroup to another. The subcategory of ribbons $\mathrm{Ribb}(S)$ has the same objects as $\mathrm{Conj}(S)$, but the morphisms are only the ones that can be constructed as ribbons—see the next section for a precise definition of $\mathrm{Conj}(S)$ and $\mathrm{Ribb}(S)$. Denote by $\mathrm{Conj}(S; X, Y)$ and $\mathrm{Ribb}(S; X, Y)$ the set of morphisms from the subgroup A_X to A_Y in the categories $\mathrm{Conj}(S)$ and $\mathrm{Ribb}(S)$ respectively.

Definition 3. Let (A_S, S) be an Artin–Tits system, and X, Y be two subsets of S. We say that the pair (X, Y) has *Property* $(\star\star\star)$ if $\mathrm{Conj}(S; X, Y)$ and $\mathrm{Ribb}(S; X, Y)$ coincide. We say that the group A_S has *Property* $(\star\star\star)$ if

$$\mathrm{Conj}(S) = \mathrm{Ribb}(S). \tag{$\star\star\star$}$$

In other words, the group A_S has Property $(\star\star\star)$ when all pairs (X, Y) have Property $(\star\star\star)$.

It is known that spherical type Artin–Tits groups and Artin–Tits groups of FC type have Properties $(\star)$, $(\star\star)$ and $(\star\star\star)$ [10,11].

Conjecture 1. *Every Artin–Tits group has Properties $(\star)$, $(\star\star)$ and $(\star\star\star)$.*

The Intuition that Conjecture 1 holds is supported by the fact that its monoid counterpart is true (see [10]).

The aim of this paper is to prove Properties $(\star)$, $(\star\star)$ and $(\star\star\star)$ for some Artin–Tits groups, among which are the two-dimensional Artin–Tits groups, that are neither of FC type nor of spherical type (we recall that an Artin–Tits group A_S is *two-dimensional* if for every subset X of S with cardinality at least 3, the standard parabolic subgroup A_X is not of spherical type).

Our methods are geometric, and two of our main tools are the Deligne complex, introduced in [8] and generalized in [7], and the CAT(0) theory.

Let us postpone most of the definitions to the next sections and just state the results precisely. We denote by D_S the Deligne complex of A_S. If X is a subset of S, we denote by A_X the standard parabolic subgroup of A_S generated by X. The results we prove are as follows (see Sections 2 and 4.3. for the definition of the Deligne complex and CAT(0) spaces):

Theorem 1. *Assume that (A_S, S) is an Artin–Tits system such that the Deligne complex D_S of A_S has a piecewise Euclidean* CAT(0) *geometric realization Γ_S. Let X be a subset of S such that A_X is of spherical type; then*

(i) *The subgroup A_X has Property $(\star)$.*
(ii) *Let Y be in S; assume that A_Y is of spherical type, or more generally that the geometric subcomplex Γ_Y of Γ_S associated with Y is convex. Then, the pair (X, Y) has Property $(\star\star)$.*
(iii) *For every subset Y of S such that the geometric subcomplex Γ_Y of Γ_S associated with Y is convex, the pair (X, Y) has Property $(\star\star\star)$.*

Theorem 2. *Assume that (A_S, S) is an Artin–Tits system such that the Deligne complex D_S of A_S has a piecewise Euclidean* CAT(0) *geometric realization Γ_S. Let X be a subset of S such that the geometric subcomplex Γ_X of Γ_S associated with X is convex; then we have*

$$\mathrm{Com}_{A_S}(A_X) = \mathrm{N}_{A_S}(A_X).$$

Theorem 3. *Assume that (A_S, S) is a two-dimensional Artin–Tits system. Then the group A_S has Properties $(\star)$, $(\star\star)$ and $(\star\star\star)$.*

Charney proved in [5] that, if the chosen CAT(0) realization of D_S is the Moussong one (see Section 2), then Γ_Y is convex for every subset Y of S. She proved that Γ_Y is also convex for the cubical metric on the Deligne complex for FC type Artin–Tits groups. As a consequence, Theorems 1 and 2 provide new proofs for the result that Artin–Tits groups of FC type have Properties $(\star)$, $(\star\star)$ and $(\star\star\star)$. To the best of our knowledge, these results are new for the family of Artin–Tits groups such that the Moussong realization of the Deligne complex is CAT(0). Theorem 3 seems to be related to no previous result.

The paper is organized as follows. In Section 1 we recall some properties of Artin–Tits groups. In Section 2 we introduce our geometric tools. Section 3 is devoted to the case of parabolic subgroups of spherical type. In particular we prove Theorems 1 and 2 (Corollaries 3.5, 3.7 and 3.11). In Section 4 we begin with the study of non-spherical type parabolic subgroups of every Artin–Tits group and, after investigating geometric properties of CAT(0) spaces, we address the case of two-dimensional Artin–Tits groups and prove Theorem 3.

1. Artin–Tits groups

We recall here some basic definitions, notation and results on Artin–Tits groups. Let S be a finite set and $M = (m_{s,t})_{s,t\in S}$ be a symmetric matrix with $m_{s,s} = 1$ for s in S and $m_{s,t}$ in $\{2, 3, 4, \ldots\} \cup \{\infty\}$ for $s \neq t$ in S. The Artin–Tits system associated with M is the pair (A_S, S) where A_S is the group defined by the presentation

$$A_S = \langle S \mid \underbrace{sts\cdots}_{m_{s,t}\text{ terms}} = \underbrace{tst\cdots}_{m_{s,t}\text{ terms}};\ \forall s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty\rangle. \tag{$*$}$$

The relations $\underbrace{sts\cdots}_{m_{s,t}\text{ terms}} = \underbrace{tst\cdots}_{m_{s,t}\text{ terms}}$ are called *braid relations* and the group A_S is said to be an *Artin–Tits group*. For instance, if $S = \{s_1, \ldots, s_n\}$ with $m_{s_i,s_j} = 3$ for $|i - j| = 1$ and $m_{s_i,s_j} = 2$ otherwise, then the associated Artin–Tits group is the braid group on $n + 1$ strings. We denote by A_S^+ the submonoid of A_S generated by S. This monoid A_S^+ has the same presentation as the group A_S, considered as a monoid presentation [14] and is cancellative. When we add the relations $s^2 = 1$ to the presentation $(*)$, we obtain the Coxeter group W_S associated with A_S. One says that A_S is of *spherical type* if W_S is finite. The matrix M may be represented by a graph, whose vertex set is S and where an edge connects two vertices if $m_{s,t} \geq 3$; these edges are labelled by $m_{s,t}$ for $m_{s,t} \geq 4$. One says that A_S (or simply S) is *indecomposable* when this graph is connected. The indecomposable components of S are the maximal subsets of S that are indecomposable.

A subgroup A_X of A_S generated by a subset X of S is called a *standard parabolic subgroup*, and a subgroup of A_S conjugated to a standard parabolic subgroup is called a parabolic subgroup. Van der Lek has shown in [16] that (A_X, X) is canonically isomorphic to the Artin–Tits system associated with the matrix $(m_{s,t})_{s,t\in X}$; its graph is the subgraph induced by X.

For every two elements a and b in the monoid A_S^+, one says that a *left-divides* b if $b = ac$ for some c in A_S^+; in that case, we write $a \prec b$. We define in the same way the right-divisibility and write $b \succ a$ if a *right-divides* b. The following result is well known:

Lemma 1.1 ([4]). *Let (A_S, S) be an Artin–Tits system. Then, the set S has a least common multiple (lcm) in A_S^+ for left-divisibility if and only if S has a least common multiple (lcm) in A_S^+ for right-divisibility if and only if A_S is of spherical type. In that case, these two lcm's are equal and are denoted by Δ_S.*

As a consequence, for every spherical type Artin–Tits group A_S and for every subset X of S, the subgroup A_X is of spherical type and Δ_X left-divides Δ_S in A_S^+.

We recall now some notation introduced in [11] and define the categories $\mathrm{Conj}(S)$ and $\mathrm{Ribb}(S)$.

Notation 1.2. Let (A_S, S) be an Artin–Tits system and X be a subset of S.

(i) We denote by X_s and X_{as} respectively the union of the spherical type indecomposable components of X and the union of the non-spherical type indecomposable components of X.

(ii) We set $X^\perp = \{s \in S; \forall t \in X,\ m_{s,t} = 2\}$ and for k in $\mathbb{N}$, $X^k = \{s^k; s \in X\}$; in particular $\emptyset^\perp = S$.

Note that we always have $X \cap X^\perp = \emptyset$ because $m_{s,s} = 1$ for every s in S. For short we will write $X_{as}^\perp$ for $(X_{as})^\perp$; this notation is unambiguous in our context.

The following definitions of the categories $\mathrm{Conj}(S)$ and $\mathrm{Ribb}(S)$ are technical. The reader may prefer to keep in mind the rough definition of the introduction and skip the precise definition.

Definition 1.3. Let (A_S, S) be an Artin–Tits system.

(i) We define the groupoid $\mathrm{Conj}(S)$ as follows: The objects of $\mathrm{Conj}(S)$ are all the subsets of S and the set $\mathrm{Conj}(S; X, Y)$ of morphisms from X to Y is in 1–1 correspondence with the set $\{g \in A_S \mid gXg^{-1} = Y\}$. The composition of morphisms is defined by the product in A_S: $g \circ f = gf$.

(ii) Let X, Y be two subsets of S; we say that an element w of $\mathrm{Conj}(S; X, Y)$ is a *positive elementary Y-ribbon-X* if either $w = \Delta_Z$ holds for some indecomposable component Z of X or there exists $t \in S$ such that the indecomposable component Z of $X \cup \{t\}$ containing t is of spherical type and $w = \Delta_Z \Delta_{Z-\{t\}}^{-1}$.

We say that an element w of $\mathrm{Conj}(S; X, Y)$ is an *elementary Y-ribbon-X* if it is a positive elementary ribbon or w^{-1} is a positive elementary X-ribbon-Y.

(iii) We denote by $\mathrm{Ribb}(S)$ the smallest subcategory of $\mathrm{Conj}(S)$ that has the same objects as $\mathrm{Conj}(S)$ and that contains the elementary ribbons; the set of morphisms from X to Y in $\mathrm{Ribb}(S)$ is denoted as $\mathrm{Ribb}(S; X, Y)$ and its elements are called *Y-ribbons-X*.

Recall that for X a subset of S, The *quasi-centralizer* $\mathrm{QZ}_{A_S}(X)$ of the subgroup A_X in A_S is

$$\mathrm{QZ}_{A_S}(X) = \{g \in A_S | gX = Xg\}.$$

Hence, by definition, we have $\mathrm{QZ}_{A_S}(X) = \mathrm{Conj}(S; X, X)$.

Proposition 1.4 ([10] Proposition 2.1). *Let (A_S, S) be an Artin–Tits system of spherical type and X, Y be two subsets of S. Let k be in $\mathbb{Z} - \{0\}$ and g be in A_S; then the following statements are equivalent:*

(1) $gA_Xg^{-1} \subseteq A_Y$;
(2) $g\Delta_X^k g^{-1} \in A_Y$;
(3) $g = yx$ *for some* $y \in A_Y$, $x \in \mathrm{Conj}(S; R, X)$ *for some* $R \subseteq Y$.

Let us finish this section with the definition of two families of Artin–Tits groups, namely the family of two-dimensional Artin–Tits groups and the family of FC type Artin–Tits groups. We do not study the latter family in this paper, but we refer to it several times, and that is why we give its definition for completeness.

Definition 1.5. Let (A_S, S) be an Artin–Tits system. Following [5], one says that (A_S, S) is a *two-dimensional Artin–Tits system* (or that A_S is a two-dimensional Artin–Tits group) if for every subset X of S with cardinality at least 3, A_X is not of spherical type.

Definition 1.6. Let (A_S, S) be an Artin–Tits system. One says that (A_S, S) is an Artin–Tits system of FC type if the following property holds: if X is a subset of S such that for all s, t in X, $m_{s,t}$ is finite, then A_X is of spherical type.

2. Deligne complex and CAT(0) realization

In this section, we introduce one of our main tools : the Deligne complex, which is a simplicial complex on which the Artin–Tits group acts. In the first subsection, we define the Deligne complex. In the second subsection, we recall the notion of a CAT(0) space. In the third subsection, we recall the construction of a particular realization of the Deligne complex, namely the Moussong realization, which is known to be CAT(0) in the case of two-dimensional Artin–Tits groups and conjecturally CAT(0) for every Artin–Tits group.

2.1. The Deligne complex

We are now going to introduce the Deligne complex and some of its geometric realizations. The Deligne complex was initially defined in [8] by Deligne in the case of spherical type Artin–Tits groups. The construction was generalized by Charney and Davis in [7]. Let (A_S, S) be an Artin–Tits group; we set

$$\mathcal{S}_{f,S} = \{T \subseteq S;\ A_T \text{ is of spherical type}\}$$

and

$$A_S\mathcal{S}_{f,S} = \{xA_T;\ x \in A_S \text{ and } T \in \mathcal{S}_{f,S}\}.$$

Recall that the complex associated with a partially ordered set $(P, \leq)$ is the abstract simplicial complex whose vertices are the elements of P and where a finite set of vertices spans a simplex if these vertices can be ordered into an increasing sequence, for the partial order $\leq$, of elements of P. The Deligne complex D_S is the complex associated with $A_S\mathcal{S}_{f,S}$ partially ordered by inclusion. Note that xA_X is a subset of yA_Y if and only if X is a subset of Y and $y^{-1}x$ is in A_Y. In order to prevent confusion between the vertex A_X and the standard parabolic subgroup A_X, we sometimes write eA_X for the vertex, where e is the unit element of A_S.

The group A_S acts by left multiplication on D_S and thus simplicially; the subcomplex K_S of D_S is the subcomplex of D_S generated by the vertices eA_T where T is in $\mathcal{S}_{f,S}$; it is a fundamental domain of D_S for the action of A_S and is the union of the K_X for X in $\mathcal{S}_{f,S}$. An interval of D_S is called an *abstract cell*; if K is an abstract cell of D_S, then it has a greatest vertex aA_X and a lowest vertex aA_Y with X, Y in $\mathcal{S}_{f,S}$ and Y a subset of X; the set of vertices of K is $\{aA_Z;\ Y \subseteq Z \subseteq X\}$ and its dimension is $\#(X - Y)$. In that case, we write $K = K(aA_Y, aA_X)$. For instance, $K_X = K(A_\emptyset, A_X)$.

One can associate a geometric realization with the abstract simplicial complex D_S. In that case, the geometric realization of an abstract cell is called a *cell*. We will only consider realizations such that A_S acts by isometries: we choose first a realization of K_S, and then we extend it to D_S using the action of A_S. The complex D_S is commonly identified with a chosen realization, even if such a realization is not unique.

If D is a set and G a group that acts on D, then the subgroups $\mathrm{Fix}(C) = \{g \in G; \forall x \in C,\ g \cdot x = x\}$ and $\mathrm{Stab}(C) = \{g \in G;\ g \cdot C = C\}$ are called the *pointwise stabilizer* and the *stabilizer*, respectively, of the subset C of D. If $C = \{x\}$, we write $\mathrm{Fix}(x)$ for $\mathrm{Fix}(\{x\}) = \mathrm{Stab}(\{x\})$. The stabilizer of a vertex aA_X of D_S is aA_Xa^{-1} and the stabilizer of cell $K = K(aA_Y, aA_X)$ is aA_Ya^{-1}; it is also its pointwise stabilizer. Let Γ_S be a realization of D_S. Since A_S acts by isometries and simplicially on Γ_S, the pointwise stabilizer of a point x is aA_Ya^{-1} if x is a vertex of the form $x = aA_Y$ or if x is in the interior of a cell $K = K(aA_Y, aA_X)$. If K_1 and K_2 are two cells of D_S then we denote by $\mathrm{span}(K_1, K_2)$ the smallest cell of D_S that contains K_1 and K_2, when it exists. Let us recall the two following results:

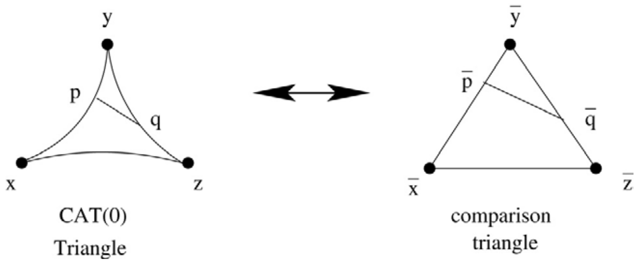
Lemma 2.1 ([16] Theorem 4.13). *Let (A_S, S) be an Artin–Tits system and X, Y be subsets of S. Then, $A_X \cap A_Y = A_{X\cap Y}$.*

Lemma 2.2 ([1] Lemma 4.1). *Let (A_S, S) be an Artin–Tits system. Let $K_1 = K(a_1A_{R_1}, a_1A_{T_1})$ and $K_2 = K(a_2A_{R_2}, a_2A_{T_2})$ be two cells of the complex D_S. Then,* $\mathrm{span}(K_1, K_2)$ *exists if and only if $T_1 \cup T_2 \in \mathcal{S}_{f,S}$ and $a_1A_{R_1} \cap a_2A_{R_2} \neq \emptyset$. Furthermore, in that case, we have* $\mathrm{span}(K_1, K_2) = K(bA_{R_1\cap R_2}, bA_{T_1\cup T_2})$ *with b in $a_1A_{R_1} \cap a_2A_{R_2}$.*

In Lemma 4.1 of [1] the Artin–Tits group is assumed to be of type FC, but the statement remains true in the general context, and the proof is the same.

2.2. CAT(0) *spaces*

Since the seminal work of M. Gromov, CAT(0) spaces have become an important tool in group theory. The existence for a group of a CAT(0) space on which the group acts by isometries implies several properties for that group. In this section, we recall basic definitions and results on CAT(0) spaces. We refer the reader to [3] for more details. Let (X, d) be a metric space. A geodesic between two points x and y of X is an isometry γ from $[0; d(x, y)]$ to X such that $\gamma\{0; 1\} = \{x; y\}$. One says that this geodesic is oriented from x to y if $\gamma(0) = x$. Following [3], we identify γ with its image, which is denoted by $[x, y]$ (even though the geodesic is not unique). A geodesic triangle $\Delta(x, y, z)$ of X is the union of three geodesics $[x, y]$, $[y, z]$ and $[z, x]$ of X, and a comparison triangle is a triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ of the Euclidean plane $\mathbf{E}_2$ such that $d_{\mathbf{E}_2}(\overline{a}, \overline{b}) = d(a, b)$ for every a, b of $\{x, y, z\}$. If p is on $[x, y]$, the comparison point of p in $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ is the point $\overline{p}$ of $[\overline{x}, \overline{y}]$ such that $d_{\mathbf{E}_2}(\overline{x}, \overline{p}) = d(x, p)$. One says that the triangle $\Delta(x, y, z)$ verifies the CAT(0) hypothesis if for every p, q in $\Delta(x, y, z)$ one has $d(p, q) \leq d_{\mathbf{E}_2}(\overline{p}, \overline{q})$. The metric space X is said to be CAT(0) if every two points of X can be joined by a geodesic and every geodesic triangle of X verifies the CAT(0) hypothesis.



When a metric space is CAT(0), then numerous properties of the Euclidean spaces still hold; let us start with the key one, which follows easily from the definition (see [3] for instance):

Proposition 2.3. *If (X, d) is a* CAT(0) *metric space, then there exists a unique geodesic (up to orientation) between every two points of X.*

If (X, d) is a metric space and Y is a non-empty subspace of X, then Y is said to be *convex* if for every two points x, y in Y, every geodesic $[x, y]$ is included in Y.

If X is CAT(0), one can define the angle $\angle_x(y, z)$ between two geodesics $[x, y]$ and $[x, z]$; this angle is called the *Alexandrov angle* of the two geodesics. We have the following properties:

Proposition 2.4 (*[3] Proposition II 2.4*). *Let (X, d) be a* CAT(0) *metric space and C a complete non-empty convex subspace of X; then for every point x of X, there exists a unique point $\pi_C(x)$ of C such that*

$$d(x, \pi_C(x)) = \mathrm{Inf}\{d(x, y) \mid y \in C\}.$$

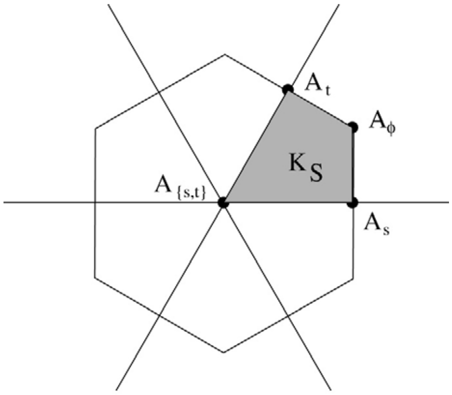
Furthermore,

(i) *if $x' \in [x, \pi_C(x)]$ then $\pi_C(x) = \pi_C(x')$;*
(ii) *if $x \notin C$ and $y \in C$ then $\angle_{\pi_C(x)}(x, y) \geq \frac{\pi}{2}$;*
(iii) *the map $x \mapsto \pi_C(x)$ is a non-decreasing retraction.*

Proposition 2.5 (*[3] II.2.11 The Flat Quadrilateral Theorem*). *Consider four distinct points p, q, r and s in a* CAT(0) *space X. Let $\alpha = \angle_p(q, s)$, $\beta = \angle_q(p, r)$, $\gamma = \angle_r(q, s)$ and $\delta = \angle_s(r, p)$. If $\alpha + \beta + \gamma + \delta \geq 2\pi$, then this sum is equal to 2π and the convex hull $[p, q, r, s]$ of the four points p, q, r and s is isometric to the convex hull of a convex quadrilateral in $\mathbf{E}_2$.*

2.3. The Moussong realization

We describe now a particular realization of the Deligne complex constructed by Charney and Davis in [7]. We follow the description given in [5]. The so-called *realization* of the Deligne complex was introduced by Moussong in his thesis [12].



S = {s,t}

Let T be in $\mathcal{S}_{f,S}$; consider a real vector space E_T of dimension $|T|$, and the standard realization of W_T as an orthogonal reflection group on E_T (see [2] Chapter V). Let C_T be the closed cone that is the fundamental domain for the action of W_T on E_T and let $x_\emptyset$ be the unique point of C_T that is at distance 1 from every wall of C_T. Consider X_T the convex hull of the W_T-orbit of $x_\emptyset$; it is a cell and it and the convex hull F_R^* of the W_R-orbit of $x_\emptyset$ for $R \subseteq T$ are faces of X_T. For s in T, denote by F_s the wall of E_T fixed by the reflection s and for $R \subseteq T$ set $F_R = \cap_{s\in R} F_s$. Then F_R and F_R^* are orthogonal and intersect in a single point x_R. The intersection of X_T and C_T is combinatorially a cube with vertices x_R for $R \subseteq T$. The Euclidean realization of K_S is obtained by choosing $X_T \cap C_T$ for realization of K_T (where x_R is the realization of A_R) for every T in $\mathcal{S}_{f,S}$; this is consistent because for $R \subseteq T$, the face F_R^* is isometric to X_R. We extend the cellular piecewise Euclidean structure of K_S to D_S using the action of A_S. This geometric realization is called the *Moussong realization* of the Deligne complex. In [7] Charney and Davis stated the following conjecture:

Conjecture 2.6 (*[7] Conjecture 4.4.4*). *The Moussong realization of every Artin–Tits group is* CAT(0).

They proved this conjecture for the family of the two-dimensional Artin–Tits groups:

Theorem 2.7. *Let (A_S, S) be a two-dimensional Artin–Tits system; then the Moussong realization of A_S is* CAT(0).

The Moussong realization is not the only interesting realization for an Artin–Tits group. For instance the Deligne complex of an Artin–Tits group of type FC has a realization that is CAT(0) ([7] Theorem 4.3.5). Some other cases are known (see [6] Corollary 5.5)

In the last section, we will use the following property:

Proposition 2.8 (*[5] Lemma 5.1*). *Let (A_S, S) be an Artin–Tits system and T be a subset of S. Denote by Γ_S and Γ_T the Moussong realizations of D_S and D_T respectively; we consider D_T (resp. Γ_T) as a subcomplex of D_S (resp. Γ_S). Assume that Γ_S is* CAT(0)*; then Γ_T is a (closed) convex subspace of Γ_S and it is* CAT(0).

3. Spherical type parabolic subgroups

The following result is the key point of our study.

Theorem 3.1. *Let (A_S, S) be an Artin–Tits system; let X, Y be two subsets of S of spherical type, let g be in A_S and k be in $\mathbb{Z} - \{0\}$. If D_S has a (piecewise Euclidean simplicial) geometric realization Γ_S that is* CAT(0) *then the following statements are equivalent:*

(1) $gA_Xg^{-1} \subseteq A_Y$;
(2) $g\Delta_X^k g^{-1} \in A_Y$;
(3) $g \in A_Y \cdot \mathrm{Ribb}(S; X, R)$ *for some* $R \subseteq Y$.

In order to prove this theorem, let us introduce the following definitions:

Definition 3.2. Let (X, d) be a CAT(0) metric space. Let C be a non-empty convex subset of X and $[x, y]$ a geodesic of X. We say that $[x, y]$ crosses C *transversally* if $C \cap [x, y]$ is a singleton and we say that $[x, y]$ crosses C internally if $[x, y] \cap C$ contains at least two distinct points.

Definition 3.3. Let (A_S, S) be an Artin–Tits system and Γ_S a realization of D_S. If x, y are two points of Γ_S, we denote by $\mathrm{span}(x)$ and $\mathrm{span}(x, y)$ respectively the smallest closed cell that contains x and the smallest closed cell that contains both x and y, if it exists. If x, y are two points of Γ_S and z is in $[x, y]$, we say that z is a transversal point of $[x, y]$ when $[x, y]$ crosses $\mathrm{span}(z)$ transversally.

Note that this definition is consistent with the notation $\mathrm{span}(K_1, K_2)$ defined in Section 2.1 above.

The following proposition is an immediate consequence of Corollary 7.29 page 110 of [3].

Proposition 3.4. *Let x, y be two points of a* CAT(0) *realization Γ_S of D_S and $\gamma : [0, d(x, y)] \to E$ be the geodesic from x to y. There exists a subdivision $0 = t_0 < t_1 < \cdots < t_n = d(x, y)$ such that for every $i \in \{1, \ldots, n-1\}$, $c_i = \gamma(t_i)$ is a transversal point of $[x, y]$ and for every $i \in \{0, \ldots, n-1\}$, the geodesic γ crosses* $\mathrm{span}(\gamma(t_i), \gamma(t_{i+1}))$ *internally. Furthermore, using convexity, one has*

$$\mathrm{span}(c_i, c_{i+1}) \cap [x, y] = [c_i, c_{i+1}].$$

Proof of Theorem 3.1. It is clear that (3) $\Rightarrow$ (1) $\Rightarrow$ (2), so it is enough to prove (2) $\Rightarrow$ (3). Recall that by Proposition 1.4 we know the implication is true when A_S is of spherical type. Assume $g\Delta_X^k g^{-1} \in A_Y$ with $k \in \mathbb{Z} - \{0\}$. It follows that the vertex $g^{-1}A_Y$ of Γ_S is fixed by Δ_X^k. Let c be a point of Γ_S and consider the geodesic γ from $e \cdot A_X$ to c in Γ_S. Set $\mathrm{span}(c) = K(hA_Z, hA_U)$; with this notation, the pointwise stabilizer of c is $hA_Z h^{-1}$. Consider the subdivision $0 = t_0 < t_1 < \cdots < t_n = d(A_X, c)$ as in Proposition 3.4. Set $c_i = \gamma(t_i)$ and $\mathrm{span}(c_i) = K(h_i A_{Z_i}, h_i A_{U_i})$. The pointwise stabilizer of c_i is then $h_i A_{Z_i} h_i^{-1}$. Let us show, by induction on i, that if c_i is fixed by Δ_X^k then $h_i^{-1} \in A_{Z_i} \cdot \mathrm{Ribb}(S; X, R_i)$ for some $R_i \subseteq Z_i$. Applying this result to $c = g^{-1}A_Y$, we will prove the theorem. Note that for $i \leq 1$, the result follows from Proposition 1.3 because we are in a cell and thus in a spherical type parabolic subgroup; so assume $i \geq 2$. Since both $e \cdot A_X$ and c are fixed by Δ_X^k, the isometry Δ_X^k fixes every point of the (unique) geodesic γ. Applying the induction hypothesis to c_{i-1}, we get that $h_{i-1}^{-1} = uv$ with u in $A_{Z_{i-1}}$ and v in $\mathrm{Ribb}(S; X, R_{i-1})$ for some subset R_{i-1} of Z_{i-1}. Now, by construction, $\mathrm{span}(c_{i-1}, c_i)$ exists and is equal to $K(\alpha A_{Z_{i-1} \cap Z_i}, \alpha A_{U_{i-1} \cup U_i})$ with $\alpha \in h_{i-1} A_{Z_{i-1}} \cap h_i A_{Z_i}$. Hence $h_i^{-1} h_{i-1} u$ is in $A_{U_{i-1} \cup U_i}$ and $(h_i^{-1} h_{i-1} u) \Delta_{R_i}^k (h_i^{-1} h_{i-1} u)^{-1}$ is equal to $h_i^{-1} \Delta_X^k h_i$ and therefore is in A_{Z_i} by assumption. Applying Proposition 1.4 in $A_{U_{i-1} \cup U_i}$, we get that $h_i^{-1} h_{i-1} u = u'v'$ with $u' \in A_{Z_i}$ and $v' \in \mathrm{Ribb}(S; R_{i-1}, R_i)$ for some $R_i \subseteq Z_i$. Thus $h_i = u'v'v$ and we are done since $v'v \in \mathrm{Ribb}(S; R_{i-1}, R_i)\mathrm{Ribb}(S; X, R_{i-1})$ is included in $\mathrm{Ribb}(S; X, R_i)$. $\square$

The two main families for which CAT(0) realizations of the Deligne complexes are known are FC type Artin–Tits groups and two-dimensional Artin–Tits groups. For both of them, every Deligne subcomplex of a parabolic subgroup is convex; hence the following corollaries apply in both cases. Note that, for FC type Artin–Tits groups these results are already known, and have been proved by a non-geometric approach (see [11]). In the case of the braid group, the first part of Corollary 3.8 has been proved in [9].

Recall that the *normalizer* $\mathrm{N}_{A_S}(A_X)$ of a standard parabolic subgroup A_X in A_S is the subgroup of A_S defined by

$$\mathrm{N}_{A_S}(A_X) = \{g \in A_S | gX \subseteq A_X g\}$$

and that an element g in A_S is in the *commensurator* $\mathrm{Com}_{A_S}(A_X)$ of A_X when $gA_Xg^{-1} \cap A_X$ has a finite index both in A_X and in gA_Xg^{-1}. Note that $\mathrm{Com}_{A_S}(A_X)$ is a subgroup of A_S. In order to state the next result, we need some notation: if G is a group and G_1, G_2 are two subgroups such that G_2 normalizes G_1, then we denote by $G_1 \cdot G_2$ the subgroup generated by G_1 and G_2. This subgroup is both $\{g_1 g_2 \in G \mid g_1 \in G_1; g_2 \in G_2\}$ and $\{g_2 g_1 \in G \mid g_1 \in G_1; g_2 \in G_2\}$.

Corollary 3.5. *Let (A_S, S) be an Artin–Tits system; let X, Y be two subsets of S of spherical type. If D_S has a (piecewise Euclidean simplicial) geometric realization Γ_S that is* CAT(0)*, then*

(i) $\mathrm{Conj}(S; X, Y) = \mathrm{Ribb}(S; X, Y)$ *and* $\mathrm{QZ}_{A_S}(X) = \mathrm{Ribb}(S; X, X)$*;*
(ii) $\mathrm{Com}_{A_S}(A_X) = \mathrm{N}_{A_S}(A_X) = A_X \cdot \mathrm{QZ}_{A_S}(X)$.

Proof. (i) By definition and Theorem 3.1 we have a sequence of inclusions

$$\mathrm{Ribb}(S; X, Y) \subseteq \mathrm{Conj}(S; X, Y) \subseteq A_Y \cdot \mathrm{Ribb}(S; X, Y).$$

It follows that $\mathrm{Conj}(S; X, Y) = \mathrm{Conj}(Y; Y, Y) \cdot \mathrm{Ribb}(S; X, Y)$. But, we also have $\mathrm{Conj}(Y; Y, Y) = \mathrm{Ribb}(Y; Y, Y) = \mathrm{QZ}_{A_Y}(Y)$ since A_Y is of spherical type. Thus $\mathrm{Ribb}(S; X, Y) = \mathrm{Conj}(S; X, Y)$.

(ii) It is clear using definition that $\mathrm{Com}_{A_S}(A_X) \supset \mathrm{N}_{A_S}(A_X) \supset A_X \cdot \mathrm{QZ}_{A_S}(X)$. Conversely, let g be in $\mathrm{Com}_{A_S}(A_X)$. Then, the set of left-cosets $\Delta_X^j(A_X \cap g^{-1}A_X g)$ with j in $\mathbb{N} - \{0\}$ in A_X is finite. Then, there exists k in $\mathbb{N} - \{0\}$ such that $g\Delta_X^k g^{-1}$ is in A_X and g is in $A_X \cdot \mathrm{QZ}_{A_S}(X)$. □

The following lemma will be crucial when proving Corollaries 3.7 and 3.10 and in the proof of Theorem 4.8.

Lemma 3.6. *Let (D, d) be a* CAT(0) *metric space and G a group that acts by isometries on D. Let C be a non-empty convex subspace of D and x a point of D; then* $\mathrm{Fix}(x) \cap \mathrm{Stab}(C) \subseteq \mathrm{Fix}(\pi_C(x))$.

Proof. One can assume that $\mathrm{Fix}(x) \cap \mathrm{Stab}(C) \neq \{0\}$. Let g be in $\mathrm{Fix}(x) \cap \mathrm{Stab}(C)$; since $g \in \mathrm{Stab}(C)$, the point $g \cdot \pi_C(x)$ is in C. Furthermore, G acts by isometries, then $d(x, \pi_C(x)) = d(g \cdot x, g \cdot \pi_C(x)) = d(x, g \cdot \pi_C(x))$. These two properties and the uniqueness part of Proposition 2.4 imply that $g \cdot \pi_C(x) = \pi_C(x)$. □

Corollary 3.7. *If in the statement of Theorem 3.1 we replace the hypothesis "Y is of spherical type" by "Γ_Y is convex in Γ_S and A_Y of non-spherical type" then the conclusion remains true. Furthermore, the subset R from (3) of Theorem 3.1 is of spherical type and* $\mathrm{Conj}(S; X, Y) = \mathrm{Ribb}(S; X, Y) = \emptyset$.

Proof. (i) As for Theorem 3.1, the implications (3) ⇒ (1) ⇒ (2) are clear. So assume (2) and let us show (3). The key idea is to prove that $g\Delta_X^k g^{-1}$ is in a spherical type parabolic subgroup of A_Y and then to apply Theorem 3.1 to conclude. Let $p = \pi_{g^{-1}\cdot\Gamma_Y}(A_X)$ be the projection of the vertex $x = eA_X$ of Γ_S on the convex subspace $g^{-1} \cdot \Gamma_Y$. As explained in Section 2.1, there exist $h \in A_Y$ and $Z \in \mathcal{S}_{f,Y}$ such that the pointwise stabilizer of p is $g^{-1}hA_Z(g^{-1}h)^{-1}$. By Lemma 3.6 we have $\mathrm{Fix}(x) \cap \mathrm{Stab}(g^{-1} \cdot \Gamma_Y) \subseteq \mathrm{Fix}(p)$. In particular, $\Delta_X^k \cdot p = p$, that is Δ_X^k is in $\mathrm{Fix}(p) = g^{-1}hA_Z(g^{-1}h)^{-1}$. By (2)⇒(3) of Theorem 3.1, we get that $g = huv$ with u in A_Z and v in $\mathrm{Ribb}(S; X, R)$ for some subset R of Z. Finally, since R is a subset of Z and Z is of spherical type, we get that R is of spherical type. The assertion $\mathrm{Conj}(S; X, Y) = \emptyset$ is consequently obvious. □

Corollary 3.8. *Let (A_S, S) be an Artin–Tits system and assume that D_S has a (piecewise Euclidean simplicial) geometric realization Γ_S that is* CAT(0).

(i) *Let s, t be in S and g be in A_S, then $gs^n g^{-1} = t^n$ for some n in $\mathbb{Z} - \{0\}$ if and only if $gsg^{-1} = t$.*
(ii) *Let s be in S, g be in A_S and Y be a subset of S such that Γ_Y is convex in Γ_S. Then, $gs^n g^{-1} \in A_Y$ for some n in $\mathbb{Z} - \{0\}$ if and only if gsg^{-1} is in A_Y.*

Proof. Point (i) is a special case of (ii). In (ii), the "if" part is clear, and the "only if" part is a consequence of (2) $\iff$ (3) of Theorem 3.1 (and Corollary 3.7) with $X = \{s\}$, $k = n$ and $k = 1$. □

Lemma 3.9. *Let (A_S, S) be an Artin–Tits system and Γ_S a* CAT(0) *realization of D_S. Let x be a point of Γ_S, g in A_S and Y a subset of S such that Γ_Y is convex and x is not in $g \cdot \Gamma_Y$. Then the endpoint $\pi_{g\cdot\Gamma_Y}(x)$ is a transversal point of the geodesic $[x, \pi_{g\cdot\Gamma_Y}(x)]$.*

Proof. Since $\mathrm{span}(\pi_{g\cdot\Gamma_Y}(x))$ is a subset of $g \cdot \Gamma_Y$ and $[x, \pi_{g\cdot\Gamma_Y}(x)]$ is a geodesic, $[x, \pi_{g\cdot\Gamma_Y}(x)]$ intersects the interior of $\mathrm{span}(\pi_{g\cdot\Gamma_Y}(x))$ in a single point $\pi_{g\cdot\Gamma_Y}(x)$. □

Corollary 3.10. *Let (A_S, S) be an Artin–Tits system; let X, Y be two subsets of S and g be in A_S; assume that D_S has a (piecewise Euclidean simplicial) geometric realization Γ_S that is* CAT(0) *and that Γ_Y is convex in Γ_S. Assume furthermore that X is maximal in S such that A_X is of spherical type. If $gA_X g^{-1}$ is a subgroup of A_Y then X is a subset of Y and g is in A_Y. If furthermore A_Y is of spherical type then $X = Y$ and eA_X is the unique point of Γ_S fixed by the subgroup A_X.*

Proof. Assume eA_X is not a vertex of $g^{-1} \cdot \Gamma_Y$. Consider the geodesic from the vertex $x = eA_X$ to $p = \pi_{g^{-1}\cdot\Gamma_Y}(A_X)$. As in Corollary 3.7, and by Lemma 3.6, the endpoint p is fixed by A_X. Since both extremities are fixed by A_X, each point of the geodesic $[x, p]$ is fixed by A_X. Let c be the first transversal point of the geodesic $[x, p]$ distinct from x; such a point exists by Lemma 3.9. Write $\mathrm{Fix}(c) = g_1 A_R$ and $\mathrm{span}(c) = K(g_1 A_R, g_1 A_T)$. By the choice of c, the points $e \cdot A_X$ and c span a cell, thus $g_1 A_R = \alpha A_R$ with $\alpha \in A_X$ and $\mathrm{span}(A_X, c) = K(\alpha A_{X\cap R}, \alpha A_{X\cup T})$. But c is fixed by A_X, hence $A_X \subseteq \alpha A_R \alpha^{-1}$. Since α is in A_X, we get that X is a subset of R; this is impossible since by maximality of X it follows that $X = R$ and $c = e \cdot A_X$. It follows that the vertex $e \cdot A_X$ is in $g^{-1} \cdot \Gamma_Y$, that is X is included in Y and g is in A_Y. When A_Y is of spherical type, we get $X = Y$ by maximality of X. $\square$

Corollary 3.11. *Let (A_S, S) be an Artin–Tits system; Assume that the Deligne complex D_S of A_S has a piecewise Euclidean geometric realization Γ_S that is* CAT(0). *Let X, Y be two subsets of S such that Γ_Y is convex in Γ_S. Then*

(i) *For every g in A_S and every $k \in \mathbb{Z} - \{0\}$, $gX^k g^{-1} \subseteq A_Y \iff gA_X g^{-1} \subseteq A_Y$ where $X^k = \{x^k \mid x \in X\}$.*
(ii) $\mathrm{Com}_{A_S}(A_X) = \mathrm{N}_{A_S}(A_X)$.

Proof. Point (i) is an immediate consequence of Corollary 3.8(ii) applied to each element of X. In (ii), By definition $\mathrm{N}_{A_S}(A_X)$ is included in $\mathrm{Com}_{A_S}(A_X)$. Conversely, let g be in $\mathrm{Com}_{A_S}(A_X)$. By the same argument as in the proof of Corollary 3.8(ii), for each s of X there exists k_s in $\mathbb{N} - \{0\}$ such that $gs^{k_s}g^{-1}$ is in A_X. Hence there exists k in $\mathbb{N} - \{0\}$ such that $gX^k g^{-1}$ is contained in A_X; by Part (i), it follows that g is in $\mathrm{N}_{A_S}(A_X)$. $\square$

4. The case of non-spherical parabolic subgroups

This section is divided into three subsections. In the first subsection, we state Conjecture 4.2, which generalizes Theorem 3.1 by removing the restriction that A_X is assumed to be of spherical type. Then, we show why, in order to prove the equivalences in the property called ($\circledast$) below, it is enough to prove them when the subgroup A_X is an indecomposable non-spherical standard parabolic subgroup. In the second subsection, we establish geometrical properties of CAT(0) spaces related to projection on convex subspaces. In the third subsection, using these geometrical properties and a little extra trick, we prove the equivalence of Property ($\circledast$)in the special case of a two-dimensional Artin–Tits group. Hence we obtain a complete proof of Theorem 3.

4.1. The main conjecture

We refer the reader back to Section 1.5 for the definition of X_s and X_{as}.

Definition 4.1. Let (A_S, S) be an Artin–Tits system. Let X, Y be two subsets of S. We say that the pair (X, Y) has property ($\circledast$) when for every g in A_S and every k in $\mathbb{Z} - \{0\}$, the following statements are equivalent:

(1) $gA_X g^{-1} \subseteq A_Y$;
(2) $g\Delta_{X_s}^k g^{-1} \in A_Y$; $X_{as}^k \subseteq A_Y$ and $g = uv$ with $u \in A_Y$ and $v \in A_{X_{as}^{\perp}}$;
(3) $X_{as} \subseteq Y$ and $g \in A_Y \cdot \mathrm{Ribb}(X_{as}^{\perp}; X_s, R)$ for some $R \subseteq Y$.

We say that the group A_S has Property ($\circledast$) when all pairs (X, Y) have Property ($\circledast$).

Theorem 3.1 suggests the following conjecture:

Conjecture 4.2. *Let (A_S, S) be an Artin–Tits system. The group A_S has Property* ($\circledast$).

In [11] we prove by algebraic methods that Artin–Tits groups of FC type have Property ($\circledast$) (with (2) slightly different). Note that (3) $\Rightarrow$ (1). If we assume that D_S has a CAT(0) realization and Γ_Y is convex, then we have (2) $\Rightarrow$ (3) using Theorem 3.1 and Corollary 3.8. Also (1) $\Rightarrow$ $g\Delta_{X_s}^k g^{-1} \in A_Y$ and $gX_{as}^k g^{-1} \subseteq A_Y$. Hence the only thing to prove under the hypothesis in Conjecture 4.2 is the implication

$$gA_X g^{-1} \subseteq A_Y \Rightarrow g = uv \quad \text{with } u \in A_Y \quad \text{and} \quad v \in A_{X_{as}^{\perp}}. \tag{†}$$

This is because in that case $X_{as}^k = vX_{as}^k v^{-1} \subseteq u^{-1} A_Y u = A_Y$.

Proposition 4.3. *Let (A_S, S) be an Artin–Tits system such that the group A_S has Property $(\circledast)$; then the group A_S has Properties $(\star)$, $(\star\star)$ and $(\star\star\star)$.*

Proof. Let (A_S, S) be an Artin–Tits system and X a subset of S. Assume A_S has Property $(\circledast)$. We have $\mathrm{Com}_{A_S}(A_X) = \mathrm{N}_{A_S}(A_X)$ as in Corollary 3.11. Statement (1) $\iff$ (3) of Property $(\circledast)$ implies $\mathrm{N}_{A_S}(A_X) = A_X \cdot \mathrm{QZ}_{A_S}(X)$. Properties $(\star\star)$ and $(\star\star\star)$ are also easy consequences of the equivalences of (1) and (3) in $(\circledast)$. $\square$

We show now that the task of proving statement $(\dagger)$ for all subsets reduces to proving it in the case when X is a particular type of standard parabolic subgroup.

Definition 4.4. Let $\mathcal{F}$ be a family of Artin–Tits systems. We say that $\mathcal{F}$ is an *SBSPS-family* if for every Artin–Tits system (A_S, S) of $\mathcal{F}$ and every subset X of S, the system (A_X, X) is in $\mathcal{F}$.

The notation "SBSPS" stands for "stable by standard parabolic subgroup". For instance, the family of Artin–Tits systems of spherical type is an SBSPS-family, the family of Artin–Tits systems of FC type is an SBSPS-family and the family of two-dimensional Artin–Tits systems is an SBSPS-family. Of course the family of all Artin–Tits systems is an SBSPS-family. Finally, by Theorem 2.7 the family of Artin–Tits systems such that the Moussong realization of their Deligne complex is CAT(0) is an SBSPS-family.

Lemma 4.5 (*First Restriction Lemma*). *Let $\mathcal{F}$ be an SBSPS-family. Assume that for every (A_S, S) of $\mathcal{F}$ and every two subsets X and Y of S such that A_X is indecomposable, $(\dagger)$ is true. Then, $(\dagger)$ is true for every (A_S, S) of $\mathcal{F}$ and every two subsets X and Y of S.*

Proof. Assume $(\dagger)$ is true for every (A_S, X, Y, g) such that (A_S, S) is in $\mathcal{F}$, g is in A_S and X and Y are two subsets of S such that A_X is indecomposable. Let X be a subset of S and $g \in A_S$ such that $gA_Xg^{-1} \subseteq A_Y$. Let us show that $(\dagger)$ is true by induction on the number n of non-spherical indecomposable components of X. If $n = 0$ the result is trivially true since $X_{as} = \emptyset$ and then $A_{X_{as}^\perp} = A_S$. If $n = 1$, it is true by hypothesis applied to X_{as} since $gA_{X_{as}}g^{-1} \subseteq gA_Xg^{-1} \subseteq A_Y$. Let $n \geq 2$. Assume the implication is true for every (A_{S_1}, X_1, Y_1, g_1) such that (A_{S_1}, S_1) is in $\mathcal{F}$, X_1 is a subset of S_1 with $n-1$ non-spherical indecomposable components, Y_1 is a subset of S_1, and g_1 is in A_{S_1}. Let X_1 be a non-spherical indecomposable component of X. We have $gA_{X_1}g^{-1} \subseteq A_Y$. Then, by hypothesis, $g = hg_1$ with $h \in A_Y$ and $g_1 \in A_{X_1^\perp}$. Now in the subgroup $A_{X_1^\perp}$ we have $g_1A_{X-X_1}g_1^{-1} \subseteq A_{Y\cap X_1^\perp}$. Since $\mathcal{F}$ is an SBSPS-family, we get, by induction hypothesis, that $g_1 = h'g_2$ with $h' \in A_Y$ and $g_2 \in A_{(X-X_1)_{as}^\perp} \cap A_{X_1^\perp}$. But $A_{(X-X_1)_{as}^\perp} \cap A_{X_1^\perp} = A_{X_{as}^\perp}$. Thus $g = hh'g_2$ with $hh' \in A_Y$ and $g_2 \in A_{X_{as}^\perp}$. $\square$

Lemma 4.6 (*Second Restriction Lemma*). *Let $\mathcal{F}$ be an SBSPS-family. Assume that for every (A_S, S) in $\mathcal{F}$, every two subsets X and Y of S, with X indecomposable with at most one element s such that $X - \{s\}$ is both indecomposable and not of spherical type and every g in A_S, $(\dagger)$ is true; then $(\dagger)$ is true for every (A_S, S) of $\mathcal{F}$, every two subsets X and Y of S and every g in A_S.*

Proof. Assume that $(\dagger)$ is true for every (A_S, X, Y, g), such that (A_S, S) is in $\mathcal{F}$, g is in A_S, and X and Y are subsets of S such that X is indecomposable with at most one element s such that $X - \{s\}$ is both indecomposable and not of spherical type. By the First Restriction Lemma, it is enough to prove that $(\dagger)$ is true when X is indecomposable and not of spherical type. Let (A_S, S) be in $\mathcal{F}$, let X and Y be in S. Assume that s and t are two distinct elements of X with the property that $X_1 = X - \{s\}$ and $X_2 = X - \{t\}$ are indecomposable and not of spherical type. Let g be in A_S such that gA_Xg^{-1} is a subgroup of A_Y. Since $gA_{X_1}g^{-1}$ is then a subgroup of A_Y we have $g = hg_1$ with h in A_Y and g_1 in $A_{X_1^\perp}$. Since $\mathcal{F}$ is an SBSPS-family, $A_{\{t\}^\perp\cup X_2}$ is in $\mathcal{F}$. As t is in X_1, the set $X_1^\perp$ is a subset of $\{t\}^\perp$ and g_1 is in $A_{\{t\}^\perp\cup X_2}$. Furthermore, in $A_{\{t\}^\perp\cup X_2}$ we have the inclusion $g_1A_{X_2}g_1^{-1} \subseteq A_{Y\cap(\{t\}^\perp\cup X_2)}$.

Thus $g_1 = h'g_2$ with h' in A_Y and g_2 in $A_{\{t\}^\perp\cup X_2} \cap A_{X_2^\perp} = A_{X^\perp}$. $\square$

4.2. Projections on Deligne subcomplexes

We prove Theorem 4.8 concerning the existence of rectangles in CAT(0) spaces and apply it in Corollary 4.9 to CAT(0) Deligne complexes. We first need a technical lemma.

Lemma 4.7. *Let (D, d) be a* CAT(0) *metric space and G a group that acts by isometries on D. Let C_1 and C_2 be two non-empty convex subspaces of D and let x be in C_1. Assume that* $\mathrm{Fix}(x) \subseteq \mathrm{Stab}(C_1) \cap \mathrm{Stab}(C_2)$ *and that for every y in C_1 different from x, the subgroup* $\mathrm{Fix}(x)$ *is not a subgroup of* $\mathrm{Fix}(y)$*; then we have $\pi_{C_1}(\pi_{C_2}(x)) = x$.*

Proof. Applying Lemma 3.6 to x and $C = C_2$, we get the inclusion $\mathrm{Fix}(x) \subseteq \mathrm{Fix}(\pi_{C_2}(x))$. Applying again this lemma to $\pi_{C_2}(x)$ and $C = C_1$, we get that the subgroup $\mathrm{Fix}(x) \subseteq \mathrm{Fix}(\pi_{C_1}(\pi_{C_2}(x)))$. By the second hypothesis applied to $y = \pi_{C_1}(\pi_{C_2}(x))$ we get $x = \pi_{C_1}(\pi_{C_2}(x))$. □

Theorem 4.8 (*Rectangle Theorem*). *Let (D, d) be a* CAT(0) *metric space and G a group that acts by isometries on D. Let C_1 and C_2 be two non-empty convex subspaces of D. Assume that x and y are two distinct points of C_1 not in C_2 and such that*

(1) $\mathrm{Fix}(x) \subseteq \mathrm{Stab}(C_1) \cap \mathrm{Stab}(C_2)$;
(2) $\forall z \in C_1, \mathrm{Fix}(x) \subseteq \mathrm{Fix}(z) \iff x = z$;
(3) $\mathrm{Fix}(y) \subseteq \mathrm{Stab}(C_1) \cap \mathrm{Stab}(C_2)$;
(4) $\forall z \in C_1, \mathrm{Fix}(y) \subseteq \mathrm{Fix}(z) \iff y = z$.

Then $[x, \pi_{C_2}(x), \pi_{C_2}(y), y]$ is isometric to a rectangle of the Euclidean plane $\mathbf{E}_2$.

Note that the angles $\angle_x(\pi_{C_2}(x), y)$, $\angle_{(\pi_{C_2}(x))}(x, \pi_{C_2}(y))$, $\angle_{(\pi_{C_2}(y))}(\pi_{C_2}(x), y)$ and $\angle_y(\pi_{C_2}(y), x)$ are consequently all equal to $\frac{\pi}{2}$; the projection on C_2 of the mid-point of x and y is the mid-point of $\pi_{C_2}(x)$ and $\pi_{C_2}(y)$; and we have the equality $d(x, y) = d(\pi_{C_2}(x), \pi_{C_2}(y))$.

Proof. By the previous lemma we have $x = \pi_{C_1}(\pi_{C_2}(x))$ and $y = \pi_{C_1}(\pi_{C_2}(y))$. Thus by Proposition 2.4, we get the four inequalities $\angle_{\pi_{C_2}(y)}(\pi_{C_2}(x), y) \geq \frac{\pi}{2}$; $\angle_{\pi_{C_2}(x)}(x, \pi_{C_2}(y)) \geq \frac{\pi}{2}$; $\angle_x(\pi_{C_2}(x), y) \geq \frac{\pi}{2}$ and $\angle_y(\pi_{C_2}(y), x) \geq \frac{\pi}{2}$. Hence, by Proposition 2.5, $[x, \pi_{C_2}(x), \pi_{C_2}(y), y]$ is isometric to a convex quadrilateral of $\mathbf{E}_2$ which must be a rectangle. □

Corollary 4.9. *Let (A_S, S) be an Artin–Tits system such that the Moussong realization Γ_S of its Deligne complex D_S is* CAT(0)*. Let X, Y be subsets of S and $g \in A_S$ such that $gA_Xg^{-1} \subseteq A_Y$. Assume that X is not of spherical type and that A_{X_1} and A_{X_2} are two distinct maximal standard parabolic subgroups of spherical type of A_X such that g is not in A_Y. Then,*

$$[A_{X_1}, \pi_{g^{-1}\cdot\Gamma_Y}(A_{X_1}), \pi_{g^{-1}\cdot\Gamma_Y}(A_{X_2}), A_{X_2}]$$

is isometric to a rectangle of the Euclidean plane $\mathbf{E}_2$.

Proof. In the notation of Theorem 4.8, set $C_1 = \Gamma_X$ and $C_2 = g^{-1} \cdot \Gamma_Y$, $x = e \cdot A_{X_1}$ and $y = e \cdot A_{X_2}$. Note that $\mathrm{Stab}(C_1) = A_X$ and $\mathrm{Stab}(C_2) = g^{-1}A_Yg$; it is clear that A_X is included in $\mathrm{Stab}(C_1)$. Considering the image of $e \cdot A_\emptyset$ by $g \in \mathrm{Fix}(C)$ we get that g is in A_X. We have $\mathrm{Fix}(e \cdot A_{X_1}) \cup \mathrm{Fix}(e \cdot A_{X_2}) = A_{X_1} \cup A_{X_2} \subseteq A_X \subseteq A_X \cap g^{-1}A_Yg = \mathrm{Stab}(C_1) \cap \mathrm{Stab}(C_2)$. Furthermore hypotheses (2) and (4) of Theorem 4.8 are a consequence of the maximality assumption. Note that neither A_{X_1} or A_{X_2} are in $g^{-1} \cdot \Gamma_Y$, since g is not in A_Y. Hence we can apply Theorem 4.8. □

4.3. *The case of two-dimensional Artin–Tits groups*

Let us recall that (A_S, S) is a two-dimensional Artin–Tits system if every standard parabolic subgroup A_X of spherical type satisfies $\#X \leq 2$. For this section, we fix a two-dimensional Artin–Tits system (A_S, S). We identify the Deligne complex Γ_S with its Moussong realization. We fix X, Y in S and g in A_S such that $gA_Xg^{-1} \subseteq A_Y$. Our objective, in this section, is to prove (†) under these hypotheses. By the first and second restriction lemmas, namely Lemmas 4.5 and 4.6, we can assume that X is indecomposable with at most one s such that $X - \{s\}$ is indecomposable and not of spherical type. In our particular case, this implies that $\#X \leq 3$ with at most one pair $\{s, t\}$ such that $m_{s,t} = \infty$. If X contains a maximal standard parabolic subgroup of spherical type, then by Corollary 3.10, the element g is in A_Y are we are done. So we assume that X does not contain such a subgroup. With the previous restrictions and the two-dimensional assumption, this means that $X = \{s, t\}$ with $m_{s,t} = \infty$.

If u is in S, we write, for short, A_u for $A_{\{u\}}$.

Proposition 4.10. *Let (A_S, S) be a two-dimensional Artin–Tits system. Let s, t be in S such that $m_{s,t} = \infty$; set $X = \{s, t\}$. Let Y be a subset of S and g in A_S such that gA_Xg^{-1} is included in A_Y. Then $g = uv$ with $v \in A_{X^\perp}$ and $u \in A_Y$. As a consequence, X is a subset of Y.*

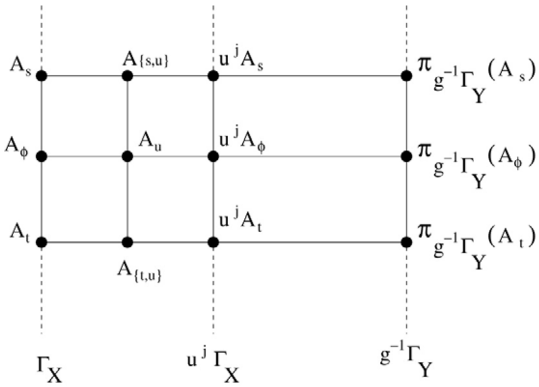
In order to prove this proposition we use the following lemma.

Lemma 4.11. *Let u, v be in S such that $m_{u,v}$ is finite, that is $\{u, v\} \in \mathcal{S}_{f,S}$. Then, we have $\angle_{e\cdot A_\emptyset}(e\cdot A_u, e\cdot A_v) \in [\frac{\pi}{2}; \pi[$ in the Moussong realization of the Deligne complex Γ_S. Furthermore, we have $\angle_{e\cdot A_\emptyset}(e \cdot A_u, e \cdot A_v) = \frac{\pi}{2}$ if and only if $m_{u,v} = 2$.*

Proof. By construction $K(e \cdot A_\emptyset, e \cdot A_{\{u,v\}})$ is a quadrilateral of $\mathbf{E_2}$ with two right angles at the vertices A_u and A_v. But the angle $\angle_{e\cdot A_{\{u,v\}}}(e \cdot A_u, e \cdot A_v)$ is acute; thus the angle $\angle_{e\cdot A_\emptyset}(e \cdot A_u, e \cdot A_v)$ is in $[\frac{\pi}{2}; \pi[$. Furthermore, $\angle_{e\cdot A_\emptyset}(e \cdot A_u, e \cdot A_v) = \frac{\pi}{2}$ if and only if $\angle_{e\cdot A_{\{u,v\}}}(e \cdot A_u, e \cdot A_v) = \frac{\pi}{2}$, that is $m_{u,v} = 2$. $\square$

Proof of Proposition 4.10. We prove the result using an induction on the number n of transversal points of the geodesic $[A_\emptyset, \pi_{g^{-1}\cdot\Gamma_Y}(A_\emptyset)]$. Note that $A_\emptyset$ is a transversal point and that $\pi_{g^{-1}\cdot\Gamma_Y}(A_\emptyset)$ is also a transversal point by Lemma 3.9.

The main idea is to study the rectangle $[A_s, \pi_{g^{-1}\cdot\Gamma_Y}(A_s), \pi_{g^{-1}\cdot\Gamma_Y}(A_t), A_t]$. Set $\mathrm{Fix}(\pi_{g^{-1}\cdot\Gamma_Y}(A_\emptyset)) = g^{-1}hA_Z$ with $h \in A_Y$ and $Z \subseteq Y$.



If $n = 1$, then $A_\emptyset = \pi_{g^{-1}\cdot\Gamma_Y}(A_\emptyset) = g^{-1}hA_Z$. Thus g is in A_Y and we are done. If $n = 2$, then $\mathrm{span}(A_\emptyset, \pi_{g^{-1}\cdot\Gamma_Y}(A_\emptyset))$ exists and $g^{-1}h$ is in A_Z; again g is in A_Y. So assume $n \geq 3$ and that g is not in A_Y. Also, assume that for every h in A_S, such that hA_Xh^{-1} is included in A_Y and the number of transversal points of the geodesic $[A_\emptyset, \pi_{h^{-1}\cdot\Gamma_Y}(A_\emptyset)]$ is lower than n, we can write $h = uv$ with $v \in A_{X^\perp}$ and u in A_Y. Since g is not in A_Y, by the same arguments as in the beginning of the proof, both geodesics $[A_s, \pi_{g^{-1}\cdot\Gamma_Y}(A_s)]$ and $[A_t, \pi_{g^{-1}\cdot\Gamma_Y}(A_t)]$ have at least three transversal points. Denote by $\gamma_s^{(1)}$ the first transversal point of $[A_s, \pi_{g^{-1}\cdot\Gamma_Y}(A_s)]$ distinct from A_s and set $\mathrm{span}(\gamma_s^{(1)}) = K(hA_R, hA_T)$; we have $\mathrm{Fix}(\gamma_s^{(1)}) = hA_R$. By definition of $\gamma_s^{(1)}$, $\mathrm{span}(A_s, \gamma_s^{(1)})$ exists. By Lemma 2.2, it follows that $A_s \cap hA_R \neq \emptyset$. Therefore, $hA_R = s^jA_R$ and $K(hA_R, hA_T) = K(s^jA_R, s^jA_T)$ for some $j \in \mathbb{Z}$. But the subgroup A_s fixes the points $e \cdot A_s$ and $\pi_{g^{-1}\cdot\Gamma_Y}(e \cdot A_s)$; then it fixes each point of the geodesic joining them and, in particular, it fixes $\gamma_s^{(1)}$, that is A_s is a subgroup of $s^jA_Rs^{-j}$. Hence $\mathrm{Fix}(\gamma_s^{(1)}) = A_R$ and s is in R. Since the point $\gamma_s^{(1)}$ is transversal, R cannot be equal to $\{s\}$; since A_S is a two-dimensional Artin–Tits group, $\gamma_s^{(1)} = e \cdot A_{\{s,u\}}$ for some $u \in S$ such that $m_{s,u} \neq \infty$.

Denote by $\gamma_t^{(1)}$ the first transversal point of $[e \cdot A_t, \pi_{g^{-1}\cdot\Gamma_Y}(e \cdot A_t)]$ distinct from $e \cdot A_t$. Similarly, we obtain $\gamma_t^{(1)} = e \cdot A_{\{t,v\}}$ for some v in S such that $m_{t,v} \neq \infty$. We are going to prove now that $u = v$, that $m_{u,s} = m_{u,t} = 2$, and that u is in $X^\perp$.

The geodesic triangle $[e \cdot A_\emptyset, e \cdot A_s, e \cdot A_{\{s,u\}}]$ is included in the geodesic rectangle $[e \cdot A_s, \pi_{g^{-1}\cdot\Gamma_Y}(e \cdot A_s), \pi_{g^{-1}\cdot\Gamma_Y}(e \cdot A_t), e \cdot A_t]$. Thus by construction of D_S and Γ_S, it follows that $B(e \cdot A_\emptyset, \epsilon) \cap K_{\{s,u\}}$ is a subset of $[e \cdot A_s, \pi_{g^{-1}\cdot\Gamma_Y}(A_s), \pi_{g^{-1}\cdot\Gamma_Y}(e \cdot A_t), e \cdot A_t]$ for some positive ϵ. In the same way there exists a positive ϵ' such that $B(A_\emptyset, \epsilon') \cap K_{\{t,v\}}$ is a subset of the rectangle $[A_s, \pi_{g^{-1}\cdot\Gamma_Y}(A_s), \pi_{g^{-1}\cdot\Gamma_Y}(A_t), A_t]$. Since $K_{\{s,u\}} \cap K_{\{t,v\}} = \{e \cdot A_\emptyset\}$, it follows that

$$0 \leq \angle_{e\cdot A_\emptyset}(e \cdot A_s, e \cdot A_u) + \angle_{e\cdot A_\emptyset}(e \cdot A_t, e \cdot A_v) \leq \angle_{e\cdot A_\emptyset}(e \cdot A_s, e \cdot A_t) = \pi.$$

But, by the previous lemma, both angles are at least equal to $\frac{\pi}{2}$. Hence, we have $\angle_{e\cdot A_\emptyset}(e\cdot A_s, e\cdot A_u) = \angle_{e\cdot A_\emptyset}(e\cdot A_t, e\cdot A_v) = \frac{\pi}{2}$, that is $m_{s,u} = m_{t,v} = 2$. Also, $K_{\{s,u\}}$ and $K_{\{t,v\}}$ are rectangles. Therefore $u = v$ and the first transversal point of $[e\cdot A_\emptyset, \pi_{g^{-1}\cdot\Gamma_Y}(e\cdot A_\emptyset)]$ distinct from $e\cdot A_\emptyset$ is $\gamma_\emptyset^{(1)} = A_u$, the mid-point of $e\cdot A_{\{u,s\}}$ and $e\cdot A_{\{u,t\}}$. Consider now $\gamma_s^{(2)}$, $\gamma_t^{(2)}$ and $\gamma_\emptyset^{(2)}$ the respective third transversal points of $[e\cdot A_s, \pi_{g^{-1}\cdot\Gamma_Y}(e\cdot A_s)]$, $[e\cdot A_t, \pi_{g^{-1}\cdot\Gamma_Y}(e\cdot A_t)]$ and $[e\cdot A_\emptyset, \pi_{g^{-1}\cdot\Gamma_Y}(e\cdot A_\emptyset)]$. The point $\gamma_s^{(1)}$, that is $e\cdot A_{\{s,u\}}$, and the point $\gamma_s^{(2)}$ span a cell, and $\gamma_s^{(2)}$ is fixed by A_s. Thus, by arguments similar to the ones used to prove that $\gamma_s^{(1)} = e\cdot A_{\{s,u\}}$, and that $m_{s,u} = 2$, we get $\gamma_s^{(2)} = u^j A_s$ for some $j \in \mathbb{Z} - \{0\}$. Furthermore, $u^j\cdot K_{\{s,u\}}$ and $u^j\cdot K_{\{t,u\}}$ are rectangles, then we have $\gamma_t^{(2)} = u^j A_t$ and $\gamma_\emptyset^{(2)} = u^j A_\emptyset$.

Now consider $g_1 = gu^j$. Since $u \in A_{X^\perp}$, we have $g_1 A_X g_1^{-1} = gA_X g^{-1} \subseteq A_Y$. But, A_S acts by isometries on Γ_S and u^{-j} send $u^j A_s$, $u^j A_t$, $u^j A_\emptyset$, $u^j\cdot\Gamma_X$ and $g^{-1}\cdot\Gamma_Y$ to $e\cdot A_s$, $e\cdot A_t$, $e\cdot A_\emptyset$, Γ_X and $g_1^{-1}\cdot\Gamma_Y$ respectively. Hence, the number of transversal points of $[e\cdot A_\emptyset, \pi_{g_1^{-1}\cdot\Gamma_Y}(e\cdot A_\emptyset)]$ is $n-2$ and we can apply the induction hypothesis: $g_1 = uv_1$ with $u \in A_Y$ and $v_1 \in A_{X^\perp}$. Finally $g = uv$ with $v = v_1 u^{-j} \in A_{X^\perp}$. $\square$

Corollary 4.12. *Let (A_S, S) be a two-dimensional Artin–Tits system. The group A_S has Property* $(\circledast)$.

As a corollary, we obtain Theorem 3.

Conjecture 4.2 remains open, but we expect that the method developed here can be extended to every Artin–Tits group, at least under Conjecture 2.6.

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